

Week 4 Worksheet - Trig sub

Please inform your TA if you find any error in the solutions.

Identities:

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta$$

Cases:

$$\sqrt{a^2 - x^2}$$

$$x = a \sin \theta$$

$$dx = a \cos \theta d\theta$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

$$\sqrt{x^2 - a^2}$$

$$x = a \sec \theta$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2 - a^2} = a \tan \theta$$

$$\sqrt{x^2 + a^2}$$

$$x = a \tan \theta$$

$$dx = a \sec^2 \theta d\theta$$

$$\sqrt{x^2 + a^2} = a \sec \theta$$

$$\sqrt{ax^2 + bx + c} \Rightarrow \text{Complete the square!}$$

1. We start today with a derivation of the following useful integral formula.

$$\begin{aligned}
 \int \sec \theta d\theta &= \int \sec \theta \frac{\tan \theta + \sec \theta}{\tan \theta + \sec \theta} d\theta \\
 &= \int \frac{\sec \theta \tan \theta + \sec^2 \theta}{\tan \theta + \sec \theta} d\theta && u = \tan \theta + \sec \theta \\
 &= \int \frac{du}{u} = \ln |u| + C && du = \sec^2 \theta + \sec \theta \tan \theta \\
 &= \ln |\tan \theta + \sec \theta| + C
 \end{aligned}$$

2.

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2 + 3}} &= \int \frac{\sqrt{3} \sec^2 \theta d\theta}{\sqrt{3} \sec \theta} && x = \sqrt{3} \tan \theta \\
 &= \int \sec \theta d\theta && dx = \sqrt{3} \sec^2 \theta d\theta \\
 &= \ln |\tan \theta + \sec \theta| + C \\
 &= \ln \left(\frac{x}{\sqrt{3}} + \frac{\sqrt{x^2 + 3}}{\sqrt{3}} \right) + C
 \end{aligned}$$

3.

$$\begin{aligned}
 \int \sqrt{16 - x^2} dx &= \int 4 \cos \theta 4 \cos \theta d\theta && x = 4 \sin \theta \\
 &= 16 \int \cos^2 \theta d\theta = 8 \int 1 + \cos(2\theta) d\theta && dx = 4 \cos \theta \\
 &= 8 \left(\theta + \frac{\sin(2\theta)}{2} \right) + C \\
 &= 8(\theta + \sin \theta \cos \theta) + C \\
 &= 8 \left(\arcsin \frac{x}{4} + \frac{x}{4} \sqrt{1 - \left(\frac{x}{4} \right)^2} \right) + C
 \end{aligned}$$

4.

$$\begin{aligned}
 \int t\sqrt{1-t^2}dt &= -\frac{1}{2}\int u^{\frac{1}{2}}du & u = 1-t^2 \\
 &= -\frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C & du = -2tdt \Rightarrow -\frac{1}{2}du = tdt \\
 &= -\frac{(1-t^2)^{\frac{3}{2}}}{3} + C
 \end{aligned}$$

5.

$$\begin{aligned}
 \int \frac{\sqrt{x^2-4}}{x^4}dx &= \int \frac{2\tan\theta}{2^4\sec^4\theta}2\sec\theta\tan\theta\tan\theta d\theta & x = 2\sec\theta \\
 &= \frac{1}{4}\int \frac{\tan^2\theta}{\sec^3\theta}d\theta & dx = 2\sec\theta\tan\theta d\theta \\
 &= \frac{1}{4}\int \sin^2\theta\cos\theta d\theta & u = \sin\theta \\
 &= \frac{1}{4}\int u^2du & du = \cos\theta d\theta \\
 &= \frac{1}{4}\frac{u^3}{3} + C & \\
 &= \frac{1}{12}\sin^3\theta + C & \\
 &= \frac{1}{12}\left(\frac{\sqrt{x^2-4}}{x}\right)^3 + C
 \end{aligned}$$

6.

$$\begin{aligned}
 \int \frac{1}{t\ln^4(t)\sqrt{\ln^2(t)-1}}dt &= \int \frac{1}{u^4\sqrt{u^2-1}}du & u = \ln t \\
 &= \int \frac{\sec\theta\tan\theta}{\sec^4\theta\tan\theta}d\theta & du = \frac{1}{t}dt \\
 &= \int \cos^3\theta d\theta & u = \sec\theta \\
 &= \int 1-v^2dv & du = \sec\theta\tan\theta d\theta \\
 &= v - \frac{v^3}{3} + C & v = \sin\theta \\
 &= \frac{\sqrt{\ln^2t-1}}{\ln t} - \frac{1}{3}\left(\frac{\sqrt{\ln^2t-1}}{\ln t}\right)^3 + C & dv = \cos\theta d\theta
 \end{aligned}$$

7.

$$\begin{aligned}\int \sqrt{e^{2x} - 1} dx &= \int \frac{\sqrt{u^2 - 1}}{u} du && u = e^x, \quad du = e^x dx \\ &= \int \frac{\tan \theta}{\sec \theta} \sec \theta \tan \theta d\theta && \Rightarrow dx = \frac{du}{e^x} = \frac{du}{u} \\ &= \int \tan^2 \theta d\theta && u = \sec \theta \\ &= \int 1 - \sec^2 \theta d\theta && du = \sec \theta \tan \theta d\theta \\ &= \theta - \tan \theta + C \\ &= \operatorname{arcsec}(e^x) - \sqrt{e^{2x} - 1} + C\end{aligned}$$